Markov Chains from Jeu de Taquin

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Outline

1. Tsetlin Library
2. Jeu de Taquin
3. Promotion on posets
4. Markov chain
5. Stationary distribution
6. Eigenvalues
7. Proof ideas

A model of a library

- n books on a shelf

$$\boxed{B_1}\boxed{B_2}\cdots\boxed{B_n}$$

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- n books on a shelf

$$\boxed{B_1}\boxed{B_2}\cdots\boxed{B_n}$$

- The probability of choosing book B_i is x_i.
- Once the book is chosen, it is moved to the back.

$$\boxed{B_1}\boxed{B_2}\cdots\boxed{B_i}\cdots\boxed{B_n} \to \boxed{B_1}\boxed{B_2}\cdots\boxed{B_n}\boxed{B_i} \text{ with probability } x_i.$$

A Markov chain on permutations

- Let $\pi \in S_n$ be a permutation.
- The stationary distribution is given by [Hendricks '72]

$$\mathbb{P}(\pi) = \prod_{i=1}^{n} \frac{x_{\pi_i}}{x_{\pi_1} + \cdots + x_{\pi_i}}.$$

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- A *derangement* is a permutation with no fixed points.
- d_m be the number of derangements in S_m.
- Let M_n be the transition matrix or generator. Then [Phatarfod '91]

$$\det(M_n - \lambda \mathbb{1}) = \prod_{S \subset [n]} (\lambda + x_S)^{d_{|S|}}$$

where $x_S = \sum_{i \in S} x_i$.

Example

- The case of $n = 3$:

$$M_3 = \begin{pmatrix} * & x_3 & 0 & 0 & x_3 & 0 \\ x_2 & * & x_2 & 0 & 0 & 0 \\ 0 & 0 & * & x_3 & 0 & x_3 \\ x_1 & 0 & x_1 & * & 0 & 0 \\ 0 & 0 & 0 & x_2 & * & x_2 \\ 0 & x_1 & 0 & 0 & x_1 & * \end{pmatrix} \begin{bmatrix} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{bmatrix}$$

- $$\mathbb{P}(231) = \frac{x_3 x_1}{(x_2 + x_3)(x_1 + x_2 + x_3)}$$

- Eigenvalues: 0, $-x_1 - x_2$, $-x_1 - x_3$, $-x_2 - x_3$ and $-x_1 - x_2 - x_3$ twice.

Generalizations

- Umpteen generalizations!
- Different moves, more shelves.
- Infinite libraries.
- Hyperplane arrangements [Bidigare, Hanlon, Rockmore '99]
- Left regular bands (monoids) [Brown '00]

Standard Young Tableaux

- A Young diagram is a representation of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where the entries are weakly decreasing.
- Originally defined by Schützenberger on skew tableaux
- Bijection on SYT

1	3	4	7
2	5		
6	8		

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$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 7 \\ \cline{1-2} 5 & 8 \\ \cline{1-2}\end{array} = \partial_1 \left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 8 \\ \cline{1-2}\end{array} \right)$$

A Minor modification - ∂_j

- First used by Edelman, Hibi and Stanley.
- Instead of starting by removing 1, we can remove integer j.
- Continue the procedure the same way.
- Add $n + 1$ at the end
- Subtract 1 from everything larger than j.

Example

- $$\partial_4 \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 7 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} \end{array} \right)$$

Example

1	2	3	6
·	7		
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Example

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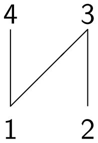
Posets

- P - a partially ordered set with order $\prec$
- $|P| = n$, "naturally" labeled by integers in $[n]$.
- $\mathcal{L}(P)$ - linear extensions of P, ways of arranging elements of P in a line respecting the order

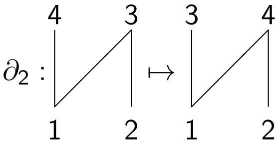
$$\mathcal{L}(P) = \{\pi \in S_n : i \prec j \Rightarrow \pi_i^{-1} < \pi_j^{-1}\} \ni e$$

- Eg: $P =$ 

, $\mathcal{L}(P) = \{1234, 1243, 1423, 2134, 2143\}$.

Jeu de Taquin (aka Promotion)

- The action of ∂_i is exactly as before.
- For example,



- Can be used to define a Markov Chain

Relation with Transposition

- Define τ_i on $\mathcal{L}(P)$

$$\pi\tau_i = \begin{cases} \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i \cdots \pi_n & \text{if } \pi_i \text{ and } \pi_{i+1} \text{ are not comparable in } P, \\ \pi_1 \cdots \pi_n & \text{otherwise.} \end{cases}$$

- Then [Haiman '92, Malvenuto & Reutenauer '94]

$$\partial_j = \tau_n \tau_{n-1} \cdots \tau_j.$$

The Directed Graph

- Given P, let G be the graph whose vertex set is $\mathcal{L}(P)$
- There is an edge $\pi \to \pi'$ if $\pi' = \partial_j(\pi)$ for some j.

Lemma

G is strongly connected.

- We will define two Markov chains on this underlying graph

Uniform Promotion Graph

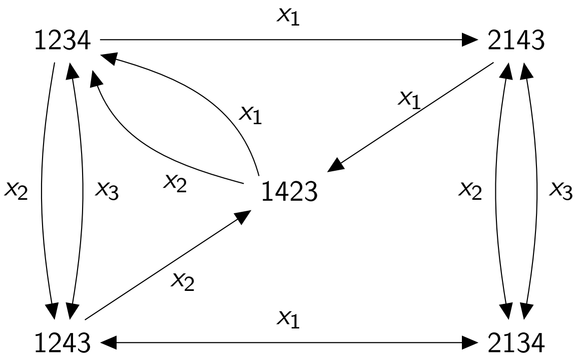
- The edge $\pi \to \pi'$, where $\pi' = \partial_j(\pi)$ has weight x_j.
- In this Markov chain, we give a probability distribution to the ∂_i's.

Theorem

The stationary distribution of the Markov chain is uniform.

- Follows from the fact that $\partial_i^k = \partial_i$ for large enough k.

Example



Promotion Graph

- The edge $\pi \to \pi'$, where $\pi' = \partial_j(\pi)$ has weight $x_{\pi(j)}$.
- In this Markov chain, we give a probability distribution to the values of the current state π.
- The stationary distribution of this Markov chain is no longer uniform.

Stationary Distribution

Theorem (1)

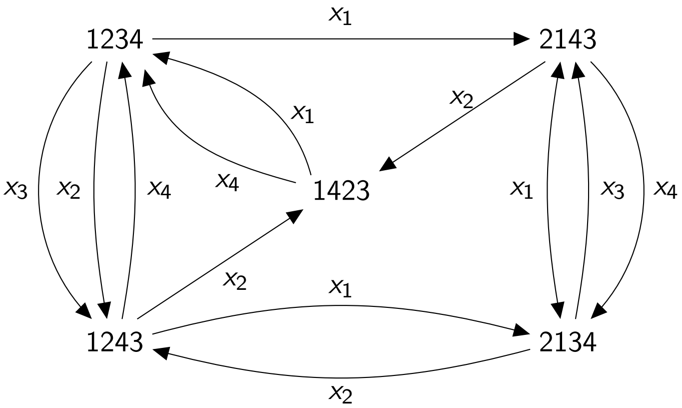
The stationary state weight $w(\pi)$ of the linear extension $\pi \in \mathcal{L}(P)$ for the continuous time Markov chain for the promotion graph is given by

$$w(\pi) = \prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}} ,$$

assuming $w(e) = 1$.

Note that $w(\pi)$ is independent of P.
Proved by induction.

Example



Example (contd.)

The transition matrix this time is given by

$$\begin{pmatrix} * & x_4 & x_1 + x_4 & 0 & 0 \\ x_2 + x_3 & * & 0 & x_2 & 0 \\ 0 & x_2 & * & 0 & x_2 \\ 0 & x_1 & 0 & * & x_1 + x_4 \\ x_1 & 0 & 0 & x_1 + x_3 & * \end{pmatrix}$$

Notice that row sums are no longer zero. The stationary distribution is

$$\left(1, \quad \frac{x_1 + x_2 + x_3}{x_1 + x_2 + x_4}, \quad \frac{(x_1 + x_2)(x_1 + x_2 + x_3)}{(x_1 + x_2)(x_1 + x_2 + x_4)}, \quad \frac{x_1}{x_2}, \quad \frac{x_1(x_1 + x_2 + x_3)}{x_2(x_1 + x_2 + x_4)}\right).$$

Recall Tsetlin library!

Special Posets

- A **rooted tree** is a connected poset, where each node has at most one successor.
- A **rooted forest** is a union of rooted trees.
- A **chain** is a totally ordered set.
- A **union of chains** is also a rooted forest.

Rooted Forests

- Note that $\sum_\pi w(\pi) \neq 1$ in general.
- The **partition function** Z_P is the prefactor that makes $\mathbb{P}(\pi) = w(\pi)/Z_P$ a probability distribution.

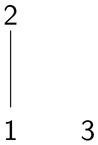
Theorem (2)

Let P be a rooted forest of size n and let $x_{\preceq i} = \sum_{j \preceq i} x_j$. The partition function for the promotion graph is given by

$$Z_P = \prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{x_{\preceq i}}.$$

Example

- $\mathcal{L}(P) = \{123, 132, 312\}$

$P =$ 

- $$w(123) = 1,\ w(132) = \frac{(x_1 + x_2)}{(x_1 + x_3)},\ w(312) = \frac{x_1(x_1 + x_2)}{x_3(x_1 + x_3)}$$

- $Z_P = \frac{x_1(x_1+x_2)(x_1+x_2+x_3)}{x_1(x_1+x_2)x_3}$

$$\mathbb{P}(123) = \frac{x_3}{x_1 + x_2 + x_3},\ \mathbb{P}(132) = \frac{x_3(x_1 + x_2)}{(x_1 + x_3)(x_1 + x_2 + x_3)},$$

$$\mathbb{P}(312) = \frac{x_1(x_1 + x_2)}{(x_1 + x_3)(x_1 + x_2 + x_3)}$$

More terminology

- An **upper set** S in P is a subset of $[n]$ such that if $x \in S$ and $y \succeq x$, then also $y \in S$.
- Let L be the lattice (by inclusion) of upper sets in P.
- $\mu(x, y)$ is the Möbius function for $[x, y] := \{z \in L \mid x \preceq z \preceq y\}$
- $f([y, \hat{1}])$ is the number of maximal chains in the interval $[y, \hat{1}]$.
- Brown defined, for each element $x \in L$, a **derangement number** d_x

$$d_x = \sum_{y \succeq x} \mu(x, y) f([y, \hat{1}]) \ .$$

Spectrum of the Transition Matrix

Theorem (3)

Let P be a rooted forest, M the transition matrix of the promotion graph, and $\overline{M} = M + (x_1 + x_2 + \cdots + x_n)\mathbb{1}$. Then

$$\det(\overline{M} - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} (\lambda - x_S)^{d_S},$$

where $x_S = \sum_{i \in S} x_i$ and d_S is the derangement number in the lattice L (by inclusion) of upper sets in P.

In other words, for each upper set $S \subseteq [n]$, there is an eigenvalue x_S with multiplicity d_S.

Corollary

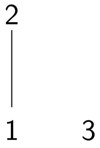
We consider the case of union of chains, and denote $P = [n_1] + [n_2] + \cdots + [n_k]$ to mean that the labels in the first chain are 1 through n_1, etc.

Lemma (4)

When P is a union of chains (labeled consecutively within chains), d_S is the number of linear extensions of $[n] \setminus S$ which are derangements.

Running Example

$P =$



- $\mathcal{L}(P) = \{123, 132, 312\}$
- Upper sets: ϕ, $\{2\}$, $\{3\}$, $\{2,3\}$, $\{1,2\}$, $\{1,2,3\}$
- Eigenvalues of $\overline{M}$: $x_1 + x_2 + x_3$, x_2, 0.

Another Example: The Tsetlin Library

- P is the n-antichain.
- $\mathcal{L}(P) = S_n$.
- Theorems 1 and 2 prove the formula about the stationary distribution.
- Theorem 3 and Lemma 4 proves the formula about the eigenvalues.

Definitions

- A **monoid** $\mathcal{M}$ is a set with an associative product and an identity.
- Natural **preorders** on $\mathcal{M}$:

$$x \leq_R y \text{ if } y = xu \text{ for some } u \in \mathcal{M}$$
$$x \leq_L y \text{ if } y = ux \text{ for some } u \in \mathcal{M}$$

- Equivalence classes on $\mathcal{M}$:

$$xRy \text{ if } y\mathcal{M} = x\mathcal{M}$$
$$xLy \text{ if } \mathcal{M}y = \mathcal{M}x$$

- $\mathcal{M}$ is R-trivial (L-trivial) if all R-classes (L-classes) are singletons. Equivalently, if the preorders are partial orders.

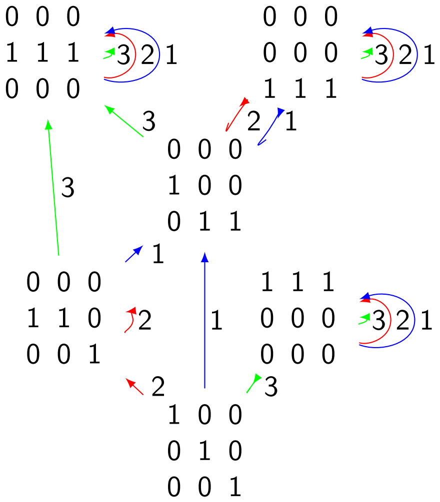
Promotion Monoid

- Define the operators G_i for $i \in [n]$ on $\mathcal{L}(P)$ by the promotion graph
$$G_i = \overline{M}_{x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_n = 0}.$$
- Define the **promotion monoid** $\mathcal{M}$ as the monoid generated by the G_i's.

Lemma (5)

$\mathcal{M}$ is R-trivial.

R-trivial monoid for the running example.



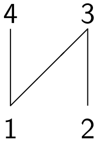
Proof ideas

- Construct the $\leq_R$ preorder on $\mathcal{M}$ and show that it is a partial order
- Prove an explicit eigenvalue formula for R-trivial monoids in general. This borrows ideas from Brown. Steinberg has such results for more general classes. This results in Theorem 3.
- Use that formula in Lemma 5 to describe degeneracies in terms of derangements. This proves Lemma 4.

Covering Times

- Recall that the uniform promotion graph led to the uniform distribution on linear extensions
- Counting linear extensions in an important problem in practice.
- Can we get better bounds on cover times? Or mixing times?
- One reason this is plausible is because the Markov chains are irreversible.

More classes of posets

- In general, the characteristic polynomial of the transition matrix does not factorize.
- But there are other classes for which it does.
- Eg: $P =$ 

$$0,\ -x_1-x_2,\ -x_1-x_2-x_4,\ -x_1-x_2-x_3-x_4,\ -2x_1-x_2-x_3-x_4.$$

Thank you for your attention!